Exact Two-Use Saturation and Strict Three-Use Separation for the Tetrahedral Schur Channel

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Abstract

We study the unhalved diamond distance $d_n(\lambda)$ from n parallel uses of the four-level tetrahedral dephasing channel to finite mixtures of arbitrary joint unitaries. The distance is exactly $d_1(\lambda) = d_2(\lambda) = \max(0, \lambda^* - \lambda)$, where $\lambda^* = 1 - \sqrt{3}/2$. At zero added noise, the unique closest two-use channel is the uniform twelve-element Bell-orbit mixture. For every $0 \le \lambda < \lambda^*$, we prove $d_3(\lambda) > d_2(\lambda)$. A single normalized Choi probe and an eight-outcome measurement reduce the comparison to a compact convex set of phase-sector distributions. An affine inequality on its saturated face excludes equality. The same certified gap is witnessed on any larger tensor power with a fixed 64-dimensional reference and computational-basis spectators. Separate quantitative bounds give $0.1679700724516 < d_3(0) < 0.1785393722161$. At and above threshold every tensor power is mixed-unitary. Lean verifies the exact distances, uniqueness statements, strict separation, and quantitative bounds.

1 Setup and prior work

Let $A \in M_4(\mathbb{C})$ be the Hermitian tetrahedral matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & -i \\ 1 & -i & 0 & i \\ 1 & i & -i & 0 \end{pmatrix}, \qquad A^\dagger = A, \qquad A^2 = 3I_4.$$

Write $H = A/\sqrt{3}$, $G = I_4 + H$, $P = G/2$, and $Q = I_4 - P$. The projectors P, Q each have rank two and constant diagonal $1/2$. In this vertex order and phase convention G is $B(1/3)$ in Haagerup and Musat [1, Ex. 3.2], crediting Christensen and Vesterstrøm [2]. Haagerup and Musat proved that $T_{B(s)}$ is nonfactorizable for $0 < s < 1$. Kayser, Luoma, and Strunz display the noiseless endpoint G and study the interpolation with complete dephasing [3, Eq. (4), Sec. IV].

$$\mathcal{D}_\lambda(X) = C_\lambda \odot X, \qquad C_\lambda = (1-\lambda)G + \lambda I_4, \qquad 0 \le \lambda \le 1,$$

where $\odot$ is entrywise multiplication. We use the local P-sector weight $a = 1 - \lambda/2$, the maximal single-line P-overlap $q = (2 + \sqrt{3})/4$, and the threshold $\lambda^* = 1 - \sqrt{3}/2$. Thus $2(a - q) = \lambda^* - \lambda$.

For n uses let $D = 4^n$. Each competitor is a finite convex combination $\Psi(X) = \sum_j p_j U_j X U_j^\dagger$, with arbitrary $U_j \in U(D)$. In particular, the competitor can entangle the copies. We use the unhalved diamond norm

$$\|\Delta\|_\diamond = \sup_{\substack{m \ge 1,\ X \in M_{Dm}(\mathbb{C}) \\ \|X\|_1 \le 1}} \|(\Delta \otimes \mathrm{id}_m)(X)\|_1, \qquad d_n(\lambda) = \inf_\Psi \|\mathcal{D}_\lambda^{\otimes n} - \Psi\|_\diamond.$$

*AI-assisted preprint. No external human specialist has reviewed this manuscript. Contributions are described at the end.

The trace norm is the nuclear norm, dual to the operator norm with respect to $\Re \operatorname{Tr}(Y^\dagger X)$. Define $J(\Phi)$ as the output of $\Phi \otimes \mathrm{id}_D$ on the normalized maximally entangled state $\omega_D = |\Omega_D\rangle\langle\Omega_D|$, where $|\Omega_D\rangle = D^{-1/2} \sum_i |i, i\rangle$.

Tensor powers of general unital channels can exhibit Birkhoff revivals. For example, in the qutrit Werner–Holevo family $\lambda W_3^+ + (1-\lambda) W_3^-$, one copy fails to be mixed-unitary for $\lambda < 1/3$, whereas two copies are mixed-unitary for $\lambda \geq (\sqrt{2}-1)(1-1/\sqrt{3})$ [4, Sec. V A] [5, Rem. 6.5(ii)]. Haagerup and Musat show that a product of Schur channels is mixed-unitary exactly when each factor is, and give a tensor-product distance bound with factor 1/2 [5, Thms. 2.2, 2.4]. Yu, Duan, and Xu give a factor-of-two comparison between the unrestricted and diagonal mixed-unitary distances and prove tensor monotonicity for the diagonal distance [6, Thms. 3–5]. The subsequent equality of the two Schur-channel distances [7] gives unrestricted tensor monotonicity by applying the equality to the product and its factor. The same public reduction transports a normalized Choi bound to a spectator-padded Choi probe by coherent basis copying, restriction to a diagonal spectator block, and a reference isometry. For the tetrahedral channel, we establish equality at the first two powers, uniqueness of the zero-noise two-use minimizer, and a strict increase at the third power. The padded construction identifies the eight effects and certifies the same margin on every padded probe.

Kayser, Luoma, and Strunz reported mixed-unitarity for their sampled points with $\lambda > 0.2$ [3, Sec. IV, Fig. 3]. Other results concern a different bath-coupled path [8], tensoring with a completely depolarizing ancillary factor [9], serial semigroups [10], and discrimination against an i.i.d. mixed-unitary null [11].

2 Exact distances and strict separation

Theorem 1 (One- and two-use saturation) *For every $\lambda \in [0, 1]$,*

$$d_1(\lambda) = d_2(\lambda) = \max(0, \lambda^* - \lambda).$$

For $\lambda \in [0, \lambda^*]$, $\mathcal{D}_{\lambda^*}$ is the unique one-use minimizing mixed-unitary channel. Every positive-weight unitary is, up to a scalar phase, one of the four vertex-phase diagonals, with grouped weight 1/4 per vertex.

Corollary 1 (Zero-noise two-use uniqueness) *At $\lambda = 0$, the uniform twelve-element Bell-orbit phase mixture is the unique two-use minimizing channel among finite mixtures of arbitrary unitaries. Every positive-weight unitary is one of the twelve orbit phases up to a scalar phase, with grouped weight 1/12 per orbit label. A decomposition may split weights among repeated labels.*

In one use the lifted P projector has target Choi probability a, while any unitary diagonal vector d has $\frac{1}{4} d^\dagger P d \leq q$. This gives the lower bound $2(a-q)$. Averaging the four vertex phases gives C_{λ^*}, and the diamond norm of the Schur multiplier with symbol H is at most one. This establishes the matching subthreshold upper bound. For two uses, the same lower bound follows from a one-use Choi probe with a computational-basis spectator, measuring P on the correlated subspace and projecting the spectator back onto its input. The Bell orbit is obtained by acting on the first factor of the row-major vectorization of $(I_4 + A)R$ with the twelve tetrahedral phase-monomial rotations preserving A, where R swaps basis labels $2, 3$ in zero-based indexing. The averaged Bell symbol has sector weights $(q, 0, 0, 1-q)$ on (PP, PQ, QP, QQ).

Below threshold, put $b = \lambda/2$. Mix the sixteen product threshold phases with total weight $w = (1-b)b/[q(1-q)]$ and the twelve Bell phases with total weight $1-w$, uniformly within each ensemble. The mixture's two cross-sector weights equal the target's $b(1-b)$. The remaining symbol difference is $4(a-q)(P \otimes P - Q \otimes Q)$. Each of $4(P \otimes P)$ and $4(Q \otimes Q)$ defines a channel, giving the diamond upper bound $2(a-q)$. At and above threshold the one-use channel is mixed-unitary by interpolation with complete dephasing. Taking its tensor square gives $d_2(\lambda) = 0$.

Theorem 2 (Strict three-use separation) *For each fixed $\lambda \in [0, \lambda^*)$, there is a competitor-independent constant $c(\lambda) > \lambda^* - \lambda$ such that*

$$\|J(\mathcal{D}_\lambda^{\otimes 3}) - J(\Psi)\|_1 \geq c(\lambda)$$

for every finite mixture Ψ of arbitrary 64-dimensional unitaries. In particular $d_3(\lambda) \geq c(\lambda) > d_2(\lambda)$.

Theorem 3 (Spectator witnesses and threshold tensoring) *Let $k \geq 0$. On $3 + k$ copies, prepare $\rho_k = \omega_{64} \otimes |0^k\rangle\langle 0^k|$ on the three active copies, the 64-dimensional reference, and the k spectator copies. For $0 \leq \lambda < \lambda^*$ and every finite mixture Ψ of arbitrary joint unitaries,*

$$\left\|[(\mathcal{D}_\lambda^{\otimes(3+k)} - \Psi) \otimes \mathrm{id}_{64}](\rho_k)\right\|_1 \geq c(\lambda) > d_2(\lambda)$$

with the same $c(\lambda)$ as in Theorem 2. Consequently $d_{3+k}(\lambda) \geq c(\lambda)$. For $\lambda \in [\lambda^*, 1]$, every $\mathcal{D}_\lambda^{\otimes n}$ has an exact mixed-unitary decomposition indexed by 8^n diagonal phase labels.

The compactness argument below gives a strict margin at each fixed noise value. The following explicit estimates use different projectors.

Theorem 4 (Quantitative three-use bounds) *Define*

$$q_3 = \max_{|z_i|=1} \tfrac{1}{64} z^\dagger P^{\otimes 3} z, \quad h = \tfrac{1+\sqrt{5/3}}{16}, \quad \ell = \tfrac{1-h}{15}, \quad m = \tfrac{\sqrt{h}+15\sqrt{\ell}}{4}.$$

The maximum is attained and satisfies $q_3 \leq q(161/162)^2 < q$ and $q_3 \leq qm^2$. At zero noise,

$$0.1679700724516 < 2(1 - qm^2) \leq d_3(0)$$
$$\leq \frac{22326979 - 10694120\sqrt{3}}{21307456} < 0.1785393722161.$$

For all $\lambda \in [0, 1]$ and all finite mixed-unitary competitors Ψ,

$$\|J(\mathcal{D}_\lambda^{\otimes 3}) - J(\Psi)\|_1 \geq \max\big(d_2(\lambda),\, 2[a^3 - q\beta^2]\big), \qquad \beta \in \{161/162, m\}.$$

The cubic branch exceeds $d_2(\lambda)$ for $0 \leq \lambda < q(1 - \beta^2)$. Its crossing against the positive linear branch solves $\lambda(1 - \lambda/2)(1 - \lambda/4) = q(1 - \beta^2)$.

The lower bounds use the lifted single-site and triple-projector tests. The zero-noise upper bounds come from explicit locally twirled phase-mixture competitors. The rational-phase seed gives the displayed bound, while a fourth-root seed gives $67/64 - \sqrt{3}/2 < 0.1808495962156$. The weaker triple-projector bound from $161/162$ gives $d_3(0) > 0.1569408439454$.

3 The measured phase-sector reduction

3.1 A single measurement of the full output

Let $\mathcal{I} = \{0, 1, 2, 3\}^3$ and $V : \mathbb{C}^{64} \to \mathbb{C}^{64} \otimes \mathbb{C}^{64}$ be $V|i\rangle = |i, i\rangle$. For $u = (u_1, u_2, u_3) \in \{0, 1\}^3$, put $P_u = P_{u_1} \otimes P_{u_2} \otimes P_{u_3}$, with $P_0 = P$ and $P_1 = Q$. The eight effects

$$E_u = V P_u V^\dagger + \tfrac{1}{8}(I_{4096} - VV^\dagger)$$

are positive and sum to the identity. The ℓ^1 distance between the measurement outcomes of two states is at most their trace-norm distance.

The target Choi state is supported on $\mathrm{ran}(V)$, has rank eight at $\lambda = 0$, and gives the product law

$$t_u(a) = a^{\#\{r:u_r=0\}}(1 - a)^{\#\{r:u_r=1\}}.$$

For an arbitrary unitary U, let $d_i = U_{ii}$. For its Choi output ket $|\psi_U\rangle$, the compressed vector is $V^\dagger|\psi_U\rangle = d/8$ and the compressed state is $dd^\dagger/64$. Hence the outcome probabilities are

$$\mathrm{Tr}(E_u J(\mathrm{Ad}_U)) = \tfrac{1}{64} d^\dagger P_u d + \tfrac{1}{8}\big(1 - \tfrac{1}{64}\|d\|_2^2\big).$$

3.2 Simultaneous recovery by flat phases

Since P_u has constant diagonal $1/8$, each probability equals $\mathrm{Tr}(P_u C(d))/64$ for $C(d) = dd^\dagger + \mathrm{diag}(1 - |d_i|^2)$. Write each d_i as the midpoint of two unit phases and choose independently and uniformly from each pair [7]. The resulting flat-phase vector z has second moment $\mathbb{E}[zz^\dagger] = C(d)$: independence yields the off-diagonal entries, and unit modulus gives the diagonal one. Thus one finite random-phase ensemble reproduces all eight probabilities simultaneously. Every mixed-unitary competitor has outcome vector in

$$K_3 = \mathrm{conv}\big\{(\tfrac{1}{64} z^\dagger P_u z)_{u\in\{0,1\}^3} : z \in (S^1)^{64}\big\}.$$

3.3 Saturation and strictness

Let $\delta = a - q > 0$. For $v \in K_3$, write $x = t(a) - v$, so $\sum_u x_u = 0$. The P-marginal of any copy averages sixteen single-line scores, each at most q. Consequently $\sum_{u:u_r=0} x_u \geq \delta$ for every r, whereas the total positive mass is $\|x\|_1/2$. If $\|x\|_1 = 2\delta$, then in every copy all positive mass must be in sector P and all negative mass in sector Q. It follows that $x = \delta(e_{000} - e_{111})$ and that all three marginals saturate. Each positive-weight phase cube realizing v must therefore saturate all 48 axis-aligned lines.

With $z_{000} = 1$, such a saturated cube has fourth-root entries. Its origin-normalized 4×4 faces belong to a list of 28 valid faces. An exhaustive Lean kernel check over compatible cube completions gives

$$3T - 2(H_{12} + H_{13} + H_{23}) + 216 \geq 0, \quad H_{ij} = z^\dagger A_i A_j z, \quad T = z^\dagger A_1 A_2 A_3 z,$$

where A_i acts by A on copy i. Saturation gives $z^\dagger A_i z = 96$ for each i. Write $p_{ij}(v) = \sum_{u:u_i=u_j=0} v_u$, $p_{123}(v) = v_{000}$, and $S = H_{12} + H_{13} + H_{23}$. The affine functional

$$L(v) = p_{123}(v) - q\sum_{i<j} p_{ij}(v) + 2q^3 + \tfrac{5\sqrt{3}}{36}\sum_{i<j}\big(p_{ij}(v) - q^2\big)$$

is nonnegative on the convex hull of saturated cubes. Indeed, expansion of $P = (I_4 + A/\sqrt{3})/2$ gives

$$\sum_{i<j} p_{ij} = \frac{192 + 192\sqrt{3} + S/3}{256}, \qquad p_{123} = \frac{64 + 96\sqrt{3} + S/3 + T\sqrt{3}/9}{512}$$

on each such cube, so $4608\sqrt{3}\, L(v) = 3T - 2S + 216 \geq 0$. The saturation candidate $v = t(a) - \delta(e_{000} - e_{111})$ instead gives

$$L(v) = \delta\big(\delta^2 + \tfrac{5\sqrt{3}}{12}\delta - \tfrac{3}{16}\big) < 0, \qquad 0 < \delta \leq 1 - q < \tfrac{1}{8}.$$

This excludes equality. The phase torus is compact, its sector map is continuous, and the convex hull of its image is compact by Carathéodory's theorem. Thus $c(\lambda) = \min_{v\in K_3} \|t(a) - v\|_1$ exists and exceeds $2\delta = d_2(\lambda)$. Measurement contraction gives Theorem 2.

3.4 The same eight outcomes with spectators

For $s = 0^k$, let J_s embed the active/reference space into spectator fiber s, and let $\Pi_s = J_s J_s^\dagger$. The effects for input $\rho_k = J_s \omega_{64} J_s^\dagger$ are

$$E_u^{(k)} = J_s E_u J_s^\dagger + \tfrac{1}{8}(I - \Pi_s).$$

The target preserves this fiber. For a joint unitary, the probabilities have the same formula with $d_i = U_{(i,s),(i,s)}$: the two uniform-complement terms together account for all leakage from the active correlated subspace. This gives the same $c(\lambda)$ for every k. The input has trace norm one, so it also gives the operational distance lower bound. At or above threshold, independent draws from the eight-label one-use decomposition give the tensor-power decomposition on every input and reference.

4 Open questions and scope

The exact flat-phase maximum q_3 and exact three-use distances below threshold remain open. The compactness proof provides a positive gap at each fixed subthreshold λ. The existence and value of $\lim_{\lambda\uparrow\lambda^*} d_3(\lambda)/d_2(\lambda)$ are undetermined here. Two-use minimizing-channel uniqueness at positive subthreshold noise remains open. A general sector-distance equality for higher tensor powers and quantitative distances to factorizable competitors are outside the scope of this paper.

Author and AI contributions

Erno Pirinen directed the research and verification standards and is responsible for the manuscript. Mesh Neon drafted the manuscript and its revisions. OpenAI Codex (Astra) led the Lean formalization. Claude Opus 5.5 independently reviewed the proof specifications and rebuilt the proof modules. Claude Fable 5.2 contributed exploratory derivations of the sector reduction.

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A Supplementary claim map

Lean 4.24.0 and mathlib revision `f897ebcf72cd16f89ab4577d0c826cd14afaafc7` were used to check the formal statements below. The listed proofs use the standard axioms `propext`, `Classical.choice`, and `Quot.sound`. Here the diamond distances quantify over all finite reference dimensions and all complex inputs of nuclear norm at most one. The padded comparison formalizes the same margin with a fixed 64-dimensional reference.

One-use distance (Theorem 1)

`FixedMarginal.radial_diamond_isLeast`
`FixedMarginal.radial_diamond_exact`

One-use minimizing-channel uniqueness (Theorem 1)

`FixedMarginal.radial_diamond_nearest_unique`

Two-use saturation (Theorem 1)

`FixedMarginal.collective_distance_isLeast`
`FixedMarginal.collective_distance_eq_single_use`

Two-use zero-noise uniqueness (Corollary 1)

`FixedMarginal.CollectiveNearest.collective_bell_noiseless_attains`
`FixedMarginal.CollectiveNearest.collective_noiseless_diamond_nearest_unique`

Three-use target symbol

`NamedComparison.three_use_target_at_noise`

Eight-effect measurement and Born probabilities

`ChannelEffects.effect_positive`
`ChannelEffects.effect_complete`
`ChannelEffects.target_sector_probability`
`ChannelEffects.unitary_sector_probability`
`ChannelEffects.sector_l1_le_nuclearNorm`

Simultaneous phase recovery

`FixedMarginal.PhaseMixtureRecovery.recoveredKernel_is_phase_mixture`
`FixedMarginal.PhaseMixtureRecovery.recoveredSector_is_phase_mixture`
`ChannelFinalBridge.mixed_sector_mem`

Saturated-cube enumeration and inequality

`FixedMarginal.SaturatedEquality.three_contrasts_saturate`
`FixedMarginal.saturated_tetra_phase_table`
`FixedMarginal.SaturatedCube.listedFace_complete`
`FixedMarginal.SaturatedCube.saturated_complex_moment_inequality`
`FixedMarginal.SaturatedChannelBridge.mixedPhaseSector_l1_strict`

Compactness and strict phase-sector infimum

`ChannelCompact.phaseMixtures_compact`
`ChannelCompact.tetrahedral_strict_phase_infimum`

Strict unrestricted three-use infimum (Theorem 2)

`NamedComparison.two_use_infimum_lt_three_use_infimum`
`ChannelFinalBridge.below_noiseThreshold`
`ChannelFinalBridge.below_noiseThreshold_oneUseGap`
`ChannelFinalBridge.unrestrictedDiamond_strict`
`ChannelFinalBridge.diamond_uniform_strict`
`ChannelFinalBridge.normalized_choi_uniform_strict`

Common padded probe and measurement (Theorem 3)

`ChannelPadding.tensorPower_choi_lower`
`ChannelPadding.paddedDensity_nuclearNorm`
`ChannelPadding.effect_eq`
`ChannelPadding.phase_infimum_le_padded`

Operational padded diamond infimum (Theorem 3)

`PaddedDiamond.commonMargin_le_distance`
`PaddedDiamond.commonMargin_le_inf`
`PaddedDiamond.below_threshold`
`PaddedDiamond.finValues_eq_values`
`PaddedDiamond.distance_eq_fin_sup`

Exact mixed-unitarity at and above threshold (Theorem 3)

`ThresholdTensor.threshold_tensor_mixed_unitary`
`ThresholdTensor.threshold_tensor_action`

Triple-projector maximum (Theorem 4)

`ChannelAllNoise.q3_attains_and_is_bounded`

Zero-noise three-use bounds and outward rounding (Theorem 4)

`FixedMarginal.tetrahedral_thirdUse_sharp_lower`
`FixedMarginal.ThirdUseBracket.outward_refined_lower`
`FixedMarginal.thirdUse_rational_diamond_upper`
`FixedMarginal.thirdUse_rational_upper_outward`
`FixedMarginal.thirdUse_diamond_upper`
`FixedMarginal.ThirdUseBracket.outward_bracket`

All-noise lower bounds and strict windows (Theorem 4)

`ChannelAllNoise.three_use_choi_bounds`
`ChannelAllNoise.three_use_diamond_lower`
`ChannelAllNoise.cubic_strict_above_radial`
`ChannelAllNoise.original_strict_window`
`ChannelAllNoise.sharp_strict_window`